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Modulation instability and solitons on a cw background in an optical fiber with higher-order effects

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We construct the Lax pair for a higher-order nonlinear Schrödinger equation that includes terms accounting for the third-order dispersion, the self-steepening effect, and the delayed nonlinear response effect. Two exact analytic solutions that describe (i) modulation instability and (ii) soliton propagation on a continuous wave background are obtained by using the Darboux transformation. In addition, we analyze the amplification-absorption and quintic nonlinearity effects on the second solution in the adiabatic approximation.

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I. INTRODUCTION

The propagation of ultrashort pulses in optical fibers in the form of optical solitons is receiving growing attention with a view to much potential application of solitons in long-distance communications, optical switching devices, and pulse shaping in laser sources. The possibility of compensating for the temporal broadening of a short pulse in the anomalous-dispersion regime of fibers by using nonlinearity (thus forming a so-called bright soliton) was first pointed out by Hasegawa and Tappert in 1973 [1]. This prediction was subsequently confirmed by several experiments [2]. At the same time Hasegawa and Tappert proposed that in the normal dispersion regime of the fiber, dark solitons might propagate in the form of dips embedded in a continuous-wave (cw) background [3].

The mathematical description of these solutions is specified by solving the nonlinear Schrödinger (NLS) equation

$$i\frac{\partial q}{\partial t}+\varepsilon\frac{\partial^2 q}{\partial x^2}+2|q|^2q=0,\quad \varepsilon=\pm 1, \tag{1}$$

by the inverse-scattering transform method with vanishing [4] and nonvanishing [5] boundary conditions for the anomalous ($\varepsilon=1$) and normal ($\varepsilon=-1$) dispersion regimes, respectively.

Kawata and Inoue [6] has discussed Eq. (1) under nonvanishing boundary conditions in the anomalous-dispersion regime ($\varepsilon=1$) by employing the inverse-scattering transform scheme. As a particular result, they obtained an exact solution that describes the evolution of one soliton on a cw background. Subsequently, Ma [7] derived a special case of a more general solution by using the inverse-scattering technique and discussed the two-soliton interaction. Later, Akhmediev *et al.* [8] and Adachihara *et al.* [9] also calculated this solution by using two different direct integration methods. The first was based on an algebraic ansatz, and the second used Bäcklund transformation. On the other hand, Hasegawa and Kodama [10] has already numerically analyzed the influence of a cw background on the behavior of a soliton pulse. They have observed that when the cw background was in phase with the solitonic pulse, a certain pulse compression (amplification) was achieved. Akhmediev and Wabnitz [11] suggested, for detecting the phase of a soliton pulse, to mix it with a cw background. More recently, N. Bélanger and P.-A. Bélanger [12] obtained an exact analytical expression for *n*-bright solitons on a cw background by using the Hirota method and discussed the two-soliton interaction.

However, it should be noted that all of these discussions are based on the NLS equation. If optical pulses are shorter, the standard NLS equation becomes inadequate. Some higher-order effects, such as third-order dispersion, self-steepening, and nonlinear response effects, will play important roles in the propagation of optical pulses. In order to understand such phenomena, Kodama and Hasegawa [13,14] proposed a higher-order nonlinear Schrödinger (HNLS) equation

$$\frac{\partial q}{\partial t}=i\left(\alpha_1\frac{\partial^2 q}{\partial x^2}+\alpha_2|q|^2q\right)+\alpha_3\frac{\partial^3 q}{\partial x^3}+\alpha_4\frac{\partial(|q|^2q)}{\partial x}+\alpha_5 q\frac{\partial|q|^2}{\partial x}, \tag{2}$$

where q is the slowly varying envelope of the pulse, $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, and α_5 are the real parameters related to group velocity dispersion (GVD), self-phase modulation (SPM), third-order dispersion (TOD), self-steepening, and delayed nonlinear response effect, respectively.

In recent years many authors have analyzed the HNLS equation from different points of view (e.g., Painlevé analysis, Hirota direct method, Ablowitz-Kaup-Newel-Segur (AKNS) method, inverse-scattering transform, Bäcklund transform, and conservation laws) and there have been many literatures giving the bright soliton [15–22] solution and dark soliton [23–25] solution for HNLS equation. Particularly, there have recently been several articles giving *W*-shaped solitary wave solution in the HNLS equation [26,27]. However, for all bright soliton or solitary wave solutions mentioned above, they are solved under the vanishing

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boundary conditions. How to find the exact and new-type solutions for an HNLS equation under the nonvanishing boundary conditions is an interesting work. Such an attempt appears in this paper.

This paper is organized as follows. In Sec. II, we first follow the AKNS formalism to extend the Lax pair for HNLS (Hirota type) equation to more general form by introducing a real parameter μ. And fundamental Darboux transformation [28,29] of the equation is presented on the basis of this Lax pair. In Sec. III, two exact solutions that describe (i) modulation instability and (ii) soliton propagation on a continuous wave background are given by using Darboux transformation. And we show how the higher-order terms influence these two solutions. In Sec. IV, we analyze how the amplification-absorption and quintic nonlinearity effects affect the second solution in the adiabatic approximation. The main results are summarized in Sec. V.

II. LAX PAIR FOR THE HNLS EQUATION AND ITS DARBOUX TRANSFORMATION

By setting $\alpha_2=2\mu^2\alpha_1$, $\alpha_4=6\mu^2\alpha_3$, $\alpha_4+\alpha_5=0$ in Eq. (2), we get an integrable Hirota equation as follows [30]:

$$\frac{\partial q}{\partial t}=i\left(\alpha_1\frac{\partial^2 q}{\partial x^2}+\alpha_2|q|^2q\right)+\alpha_3\frac{\partial^3 q}{\partial x^3}+\alpha_4|q|^2\frac{\partial q}{\partial x}. \quad (3)$$

Equation (3) has been investigated in the form of soliton solution with vanishing boundary conditions by several authors [29–31]. Here we concentrate on discussing it with nonvanishing boundary conditions by using Darboux transformation.

By employing the AKNS method one can construct the linear eigenvalue problem for Eq. (3) as follows:

$$\psi_x=U\psi, \quad (4)$$

$$\psi_t=V\psi, \quad (5)$$

where

$$\psi=(\psi_1,\psi_2)^T.$$

Here U and V can be given in the forms

$$U=\lambda J+P, J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad P=\begin{pmatrix}0 & -\mu q\\ \mu\bar{q} & 0\end{pmatrix}, \quad (6)$$

$$V=4\alpha_3\lambda^3\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-4\lambda^2\begin{pmatrix}-\frac{i\alpha_1}{2} & \mu\alpha_3 q\\ -\mu\alpha_3\bar{q} & \frac{i\alpha_1}{2}\end{pmatrix}+2i\lambda\begin{pmatrix}-i\mu^2\alpha_3|q|^2 & -\mu\alpha_1 q+i\mu\alpha_3 q_x\\ \mu\alpha_1\bar{q}+i\mu\alpha_3\bar{q}_x & i\mu^2\alpha_3|q|^2\end{pmatrix}$$
$$+\begin{pmatrix}i\alpha_1\mu^2|q|^2-\alpha_3\mu^2(q\bar{q}_x-\bar{q}q_x) & -i\mu\alpha_1 q_x-\mu\alpha_3 q_{xx}-2\alpha_3\mu^3 q|q|^2\\ -i\mu\alpha_1\bar{q}_x+\mu\alpha_3\bar{q}_{xx}+2\alpha_3\mu^3\bar{q}|q|^2 & -i\alpha_1\mu^2|q|^2+\alpha_3\mu^2(q\bar{q}_x-\bar{q}q_x)\end{pmatrix}, \quad (7)$$

where μ is a real constant. Obviously, setting $\alpha_3=0$ Eqs. (6) and (7) will give the Lax pair of NLS equation. The compatibility condition $U_t-V_x+[U,V]=0$ gives rise to Eq. (3). In general, the Lax pair assures the complete integrability of a nonlinear system, and is especially used to obtain an N-soliton solution by means of inverse-scattering transformation method. Here based on the Lax pair (6) and (7), we solve Eq. (3) by using the Darboux transformation and obtain two new types of solitary wave solutions by choosing the periodic initial "seed" solution.

Consider the Darboux transformation of Eq. (3),

$$\varphi'=(\lambda I-S)\varphi, S=H\Lambda H^{-1}, \Lambda=\mathrm{diag}(\lambda_1,\lambda_2), \quad (8)$$

where H is a nonsingular matrix, requiring

$$\varphi'_x=U'\varphi', \quad U'=\lambda J+P', \quad P'=\begin{pmatrix}0 & -\mu q'\\ \mu\bar{q}' & 0\end{pmatrix}. \quad (9)$$

Combining Eqs. (4), (8), and (9), we obtain the Darboux transformation for Eq. (3) in the form

$$P'=P+JS-SJ. \quad (10)$$

It is easy to verify that, if $(\varphi_1,\varphi_2)^T$ is a solution of Eqs. (4) and (5) corresponding to $\lambda=\lambda_1$, then $(-\bar{\varphi}_2,\bar{\varphi}_1)^T$ is also a solution of Eqs. (4) and (5) and the eigenvalue λ is replaced by $-\bar{\lambda}_1$, then we have

$$\Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & -\bar{\lambda}_1\end{pmatrix}, \quad H=\begin{pmatrix}\varphi_1 & -\bar{\varphi}_2\\ \varphi_2 & \bar{\varphi}_1\end{pmatrix},$$

$$\Delta=\det|H|=|\varphi_1|^2+|\varphi_2|^2,$$

$$S_{ij}=-\bar{\lambda}_1\delta_{ij}+\frac{(\lambda_1+\bar{\lambda}_1)\varphi_i\bar{\varphi}_j}{\Delta} \quad (i,j=1,2). \quad (11)$$

From Eqs. (6), (9), (10), and (11), we have

$$q' = q - \frac{2}{\mu} S_{12}, \overline{q'} = \overline{q} - \frac{2}{\mu} S_{21}. \tag{12}$$

Thus we obtain the fundamental expression of Darboux transformation. That is, if q is a solution of Eq. (3), q' is also a solution of Eq. (3). Therefore, we may view q as a ''seed.''

III. EXACT SOLUTIONS

In this section, as an example of Darboux transformation, we give two resulting explicit solutions of an HNLS equation that describe (i) modulation instability and (ii) soliton propagation on a continuous wave background.

Here we take the initial seed $q = a \exp i(At + Bx)$ and we require

$$a(A + B^2\alpha_1 + B^3\alpha_3 - Ba^2\alpha_4 - a^2\alpha_2) = 0. \tag{13}$$

To solve Eqs. (4) and (5), let $\varphi_1 = f_1 \exp i(At + Bx)$, $\varphi_2 = f_2$, then Eqs. (4) and (5) become

$$f_{1,x} + iBf_1 = \lambda f_1 - \mu a f_2,$$

$$f_{2,x} = \mu a f_1 - \lambda f_2,$$

$$iAf_1 + f_{1,t} = \alpha f_1 - \beta f_2,$$

$$f_{2,t} = \beta f_1 - \alpha f_2.$$

By solving the equations above, we obtain

$$\varphi_1 = (C_1 \exp \theta_1 + C_2 \exp \theta_2) \exp i(At + Bx),$$
$$\varphi_2 = C_3 \exp \theta_1 + C_4 \exp \theta_2, \tag{14}$$

where

$$\theta_1 = (-iB + \zeta_1 + i\eta_1)x/2 + (-iA + \zeta_2 + i\eta_2)t/2,$$

$$\theta_2 = (-iB - \zeta_1 - i\eta_1)x/2 + (-iA - \zeta_2 - i\eta_2)t/2,$$

$$\lambda = \frac{1}{2}(\lambda_1 + i\lambda_2),$$

and the expressions for real numbers ζ_1, η_1, ζ_2, and η_2 are presented in formula (17), and the relations among the constants C_1, C_2, C_3, and C_4 are given as follows:

$$C_2 = (L + iM)C_4, \quad C_3 = (L + iM)C_1.$$

Here for simplicity, we take $|C_1| = |C_4|$, where

$$L = \frac{2a\mu(\lambda_1 + \zeta_1)}{(\lambda_1 + \zeta_1)^2 + (\lambda_2 - B + \eta_1)^2},$$

$$M = \frac{-2a\mu(\lambda_2 - B + \eta_1)}{(\lambda_1 + \zeta_1)^2 + (\lambda_2 - B + \eta_1)^2}.$$

Substituting expression (14) into Eq. (12) and using formula (11), we have the following solution:

$$q = \frac{G}{F} \exp i(At + Bx), \tag{15}$$

and its corresponding nonlinear phase shift $\phi(x,t)$ is in the form

$$\phi(x,t) = \arctan\left(\frac{D_2 \sinh \Xi + D_4 \sin \Theta}{D_1 \cosh \Xi + D_3 \cos \Theta}\right), \tag{16}$$

where

$$G = D_1 \cosh \Xi + iD_2 \sinh \Xi + D_3 \cos \Theta + iD_4 \sin \Theta,$$

$$F = D_5 \cosh \Xi + D_6 \cos \Theta,$$

$$\Xi = \zeta_1 x + \zeta_2 t, \quad \Theta = \eta_1 x + \eta_2 t,$$

$$D_1 = a\mu(1 + L^2 + M^2) - 2\lambda_1 L, D_2 = 2\lambda_1 M,$$

$$D_3 = 2a\mu L - \lambda_1(1 + L^2 + M^2), \quad D_4 = -\lambda_1(1 - L^2 - M^2),$$

$$D_5 = \mu(1 + L^2 + M^2), \quad D_6 = 2\mu L, \tag{17}$$

$$\zeta_1 + i\eta_1 = \sqrt{-B^2 - 4(-\lambda^2 + a^2\mu^2 + iB\lambda)},$$

$$\zeta_2 + i\eta_2 = \sqrt{-A^2 - 4(\beta^2 - \alpha^2 + iA\alpha)},$$

$$\alpha = 4\alpha_3\lambda^3 + 2\mu^2a^2\alpha_3\lambda + i(2\alpha_1\lambda^2 + \alpha_1\mu^2a^2 + 2B\alpha_3\mu^2a^2),$$

$$\beta = 4\mu a\alpha_3\lambda^2 + 2\alpha_3\mu^3a^3 - a\mu\alpha_3B^2 - a\mu\alpha_1 B + i2\mu a\lambda(\alpha_3 B + \alpha_1).$$

Here λ_1, λ_2, ζ_1, η_1, ζ_2, η_2 are real numbers. Solution (15) has some novel properties. Here we mainly discuss three types of solitary wave solutions under the following parametric conditions.

(A) In case of $a = 0$. i.e., the initial seed is zero, solution (15) becomes the bright solution as follows:

$$q = -\frac{\lambda_1}{\mu} \operatorname{sech}\Xi \exp i\Theta, \tag{18}$$

where

$$\Xi = \lambda_1 x + \lambda_1[\alpha_3(\lambda_1^2 - 3\lambda_2^2) - 2\alpha_1\lambda_2]t,$$

$$\Theta = \lambda_2 x + [\alpha_3\lambda_2(3\lambda_1^2 - \lambda_2^2) + \alpha_1(\lambda_1^2 - \lambda_2^2)]t.$$

here $\alpha_2 = 2\mu^2\alpha_1, \alpha_4 = 6\mu^2\alpha_3$, i.e., three parameters are arbitrary among $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. The solution had been extensively discussed by other authors [30]. Here it is only an example obtained by Darboux transformation.

(B) For $a \neq 0$, the initial ''seed'' is periodic. And for simplicity, we take $\lambda_2 = B = 0$ and correspondingly have constraint condition $A = a^2\alpha_2$ from Eq. (13). And from Eq. (17), we have

$$\zeta_1 + i\eta_1 = \sqrt{\lambda_1^2 - 4a^2\mu^2}.$$

Here we should note that solution (15) is not significant for $4\mu^2a^2-\lambda_1^2=0$. Therefore, the following two cases will be investigated mainly.

(i) In the case of $4\mu^2a^2-\lambda_1^2>0$, solution (15) becomes as follows:

$$q=\frac{G}{F}\exp i(a^2\alpha_2 t),$$

$$G=D_1\cosh\Xi+iD_2\sinh\Xi+D_3\cos\Theta,$$

$$F=D_5\cosh\Xi+D_6\cos\Theta,$$

$$\Xi=\zeta_2 t,\Theta=\eta_1 x+\eta_2 t,$$

$$\eta_1^2=4a^2\mu^2-\lambda_1^2, \tag{19}$$

$$\eta_2=\alpha_3\eta_1(2a^2\mu^2+\lambda_1^2),$$

$$\zeta_1=0,\quad \zeta_2=-\alpha_1\eta_1\lambda_1,$$

$$D_1=\frac{1}{a\mu}(\eta_1^2-2a^2\mu^2),\quad D_2=-\frac{1}{a\mu}\eta_1\lambda_1,$$

$$D_3=-\lambda_1,\quad D_5=2\mu,\quad D_6=\frac{1}{a}\lambda_1.$$

By analyzing solution (19), we note that this solution is periodic in the space coordinate and aperiodic in the longitudinal variable as shown in Fig. 1. Therefore, it is considered as a modulation instability (MI) process. MI is the process by which a cw beam becomes unstable [32]. In general, a whole class of solutions of the NLS equation that are periodic or quasiperiodic both in space and time dimensions exists. The aperiodic solution in time may be viewed as a homoclinic or separatrix trajectory in the infinite-dimension phase space of the solutions of Eq. (3) with periodic boundary conditions in space [see, for example, Ref. [33] and references therein], i.e., the homoclinic orbit or separatrix trajectory is characterized by a single mode which limits to the plane wave as $t\to\pm\infty$ [34]. To our best knowledge, MI was predicted to occur in optical fibers [35] and was experimentally observed [36]. And the exact analytic expression for MI in the NLS equation was obtained [37]. However, in this paper, an exact analytic expression for MI in the HNLS equation are given. Figure 1(b) shows the propagation of this homoclinic orbit in the presence of higher-order effects. As seen from Fig. 1(a) and 1(b), when compared with the homoclinic orbit of NLS equation as given in Refs. [33,34], the main characteristics of the homoclinic orbit in the presence of higher-order terms ($\alpha_3\neq 0$) are essentially the same except for the change of group velocities. And it is interesting to find that the sign of α_3 determines the propagation direction of solitary wave. In application, the homoclinic orbit of modulation instability can be used to produce a strain of optical solitons. Another potential usage of the modulation instability effect is for code generation and decoding in code division multiaccess communication systems [33].

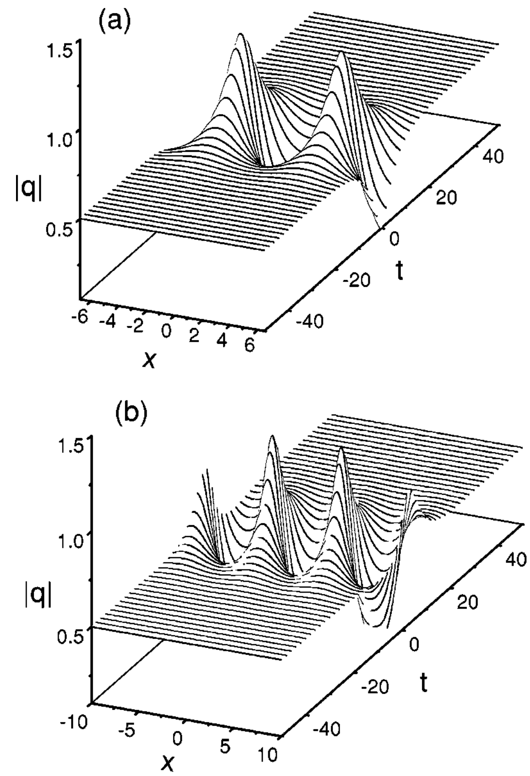

FIG. 1. Evolution of a homoclinic orbit of modulation instability with $a=0.5$, $\mu=1$, $\lambda_1=0.5$, $\alpha_1=0.5$, $\alpha_2=1$. (a) The absence of higher-order effects $\alpha_3=\alpha_4=\alpha_5=0$; (b) the presence of higher-order effects $\alpha_3=0.18$, $\alpha_4=1.08$, $\alpha_5=-1.08$.

(ii) In the case of $4\mu^2a^2-\lambda_1^2<0$, solution (15) becomes the following form:

$$q=A\exp i(a^2\alpha_2 t),$$
$$A=2\zeta_1\frac{\zeta_1\cos\Theta+i\lambda_1\sin\Theta}{\lambda_1\cosh\Xi-a\cos\Theta}-a, \tag{20}$$

where

$$\Xi=\zeta_1 x+\zeta_2 t,\Theta=\eta_2 t,$$

$$\zeta_1^2=\lambda_1^2-a^2,\zeta_2=\alpha_3\zeta_1(2a^2\mu^2+\lambda_1^2),$$

$$\eta_2=\alpha_1\zeta_1\lambda_1.$$

Figures 2(a), 2(b), and 2(c) show the propagation of such a bright soliton for different background amplitude a. As one can see from Fig. 2, solution (20) represents a bright pulse that propagates on a cw background in the presence of higher-order effects. The main characteristic of the propagation is the periodic peaking property of the field amplitude, which can be very strong without splitting of the pulse. And solution (20) with $\alpha_3=0$ of course gives such a bright soliton plus cw background for the NLS equation.

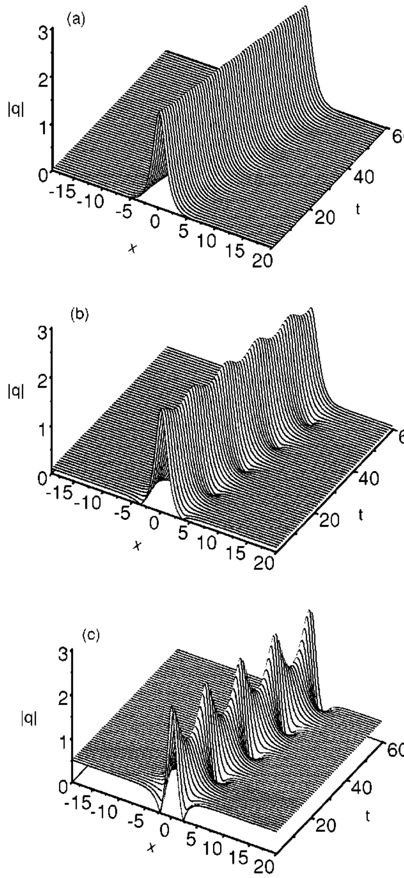

FIG. 2. Propagation of the cw soliton (20) for $\mu=0.5$, $\lambda_1=\sqrt{2}$, $\alpha_1=0.5$, $\alpha_2=0.25$, $\alpha_3=-0.10$, $\alpha_4=-0.60$, $\alpha_5=0.60$, and for three cw background amplitudes: (a) $a=\frac{1}{100}$, (b) $a=\frac{1}{10}$, (c) $a=\frac{1}{2}$.

Note that the α_3 dependence arises only through the eigenvalues ζ_2 from $[\zeta_2=\alpha_3\zeta_1(2a^2\mu^2+\lambda_1^2)]$. Thus the effect of the higher-order terms on the solitary wave solution is simply changing the coefficients of the coordinate t. This shows that, when compared with solitary wave solution of the NLS equation ($\alpha_3=0$), the main characteristics of the solution in the presence of higher-order terms are essentially the same except for the change of soliton velocities. It clearly shows the change of velocity as expected. In addition, we also find that the sign of the third-order dispersion (α_3) determines the propagation direction of solitary wave.

It is also interesting to note that

$$\int_{-\infty}^{+\infty}[|q(x,t)|^2-|q(\pm\infty,t)|^2]dx=8\zeta_1, \tag{21}$$

which is exactly the energy of the one-soliton solution of Eq. (3). In contrast, the quantity

$$\int_{-\infty}^{+\infty}|q(x,t)-q(\pm\infty,t)|^2dx=8\zeta_1+4\zeta_1 a(\cos\Theta)I, \tag{22}$$

where

$$I=\frac{4\arctan\frac{\sqrt{(\lambda_1+a\cos\Theta)}}{\sqrt{(\lambda_1-a\cos\Theta)}}}{\sqrt{\lambda_1^2-a^2\cos^2\Theta}} \tag{23}$$

shows that a t-periodic energy exchange is performed between the pulse and the cw background.

In addition, we should note that the solution takes the particular form at any location $t_0=[(1+4n)\pi/2\eta_2]$ for $n=0,1,2,\ldots$,

$$q=\exp i(a^2\alpha_2 t)(-a+2i\zeta_1\operatorname{sech}\Xi). \tag{24}$$

Therefore, this solution can be generated by coherently adding in quadrature a bright soliton to a cw background.

IV. AMPLIFICATION-ABSORPTION AND QUINTIC NONLINEARITY EFFECTS

In this section we calculate the adiabatic evolution of the parameters a, ζ_1, and λ_1 of the solution II in the presence of amplification-absorption terms and quintic nonlinearity for Eq. (3). To the end, the following equation:

$$i\frac{\partial q}{\partial t}+\left(\alpha_1\frac{\partial^2 q}{\partial x^2}+\alpha_2|q|^2q\right)-i\alpha_3\frac{\partial^3 q}{\partial x^3}-i\alpha_4|q|^2\frac{\partial q}{\partial x}=iR(q), \tag{25}$$

where

$$R(q)=\gamma_0 q+\gamma_1 q_{tt}-\gamma_2 q|q|^2-\gamma_3 q|q|^4$$

is considered.

The parameter γ_0 describes a linear amplification ($\gamma_0>0$) or absorption ($\gamma_0<0$), the parameter $\gamma_1\geqslant 0$ is a gain dispersion term that is due to a finite gain bandwidth, and $\gamma_2\geqslant 0$ stands for a phenomenological model of gain saturation or a two-photo absorption effect and γ_3 that is proportional to the fifth-order susceptibility stands for a model of gain saturation. For $\alpha_3=\alpha_4=\gamma_3=0$, we recover the equation investigated by Gagnon [38].

As usual in the adiabatic approximation, we consider that $R(q)$ is small and assume that the wave evolution is close in shape to expression (20), where the parameters a, ζ_1, and λ_1 are considered as functions of t. Therefore, here the variable

$$\frac{d}{dt}\int_{-\infty}^{\infty}|q|^2dx=2\operatorname{Re}\int_{-\infty}^{\infty}\bar{q}R(q)dx \tag{26}$$

of the first conserved integral is needed for one to determine the evolution of the parameters of the exact solution.

It is important to point out that the integrals in Eq. (26) contain a contribution that is due to the cw background. This contribution can easily be identified by taking note that the evolution of the cw part is completely determined by the parameter a and can be calculated exactly by solving Eq. (25). The result is

$$q_{cw}=a(t)\exp i\left(\int_0^t a^2\alpha_2 dt\right), \tag{27}$$

where $a(t)$ satisfies

$$\frac{da(z)}{dx}=\gamma_0 a-\gamma_2 a^3-\gamma_3 a^5. \tag{28}$$

We can eliminate the infinite cw contribution in Eq. (26) by using relation (28). The remaining terms in Eq. (26) then give the following evolution equation of $\zeta_1(z)$:

$$\frac{d\zeta_1(z)}{dx}=2\gamma_0\zeta_1-\zeta_1^3(\gamma_1+4\gamma_2)\frac{W}{\lambda_1^2-a^2\cos^2\Theta}-\frac{8}{3}\gamma_2\zeta_1^3$$
$$-4\gamma_2 a^2\zeta_1+\frac{4}{15}\gamma_3\zeta_1\lambda_1^2(12\lambda_1^2-7a^2)-\gamma_3\Delta, \tag{29}$$

where

$$W=\frac{1}{2}\lambda_1^2 a\cos\Theta I+\frac{1}{3}(2\lambda_1^2+a^2\cos^2\Theta),$$

$$\Delta=M_1+M_2+M_3,$$

$$M_1=20\lambda_1^2\frac{\zeta_1^3}{(\lambda_1^2-a^2\cos^2\Theta)^2}(\lambda_1^4-a^4\cos^2\Theta),$$

$$M_2=4a^4\frac{\zeta_1}{(\lambda_1^2-a^2\cos^2\Theta)^2}(\cos^2\Theta)(\lambda_1^4\cos^2\Theta-2\lambda_1^4$$
$$+a^4\cos^2\Theta)+12\zeta_1^5\lambda_1^4 aI\cos\Theta\frac{1}{(\lambda_1^2-a^2\cos^2\Theta)^2},$$

$$M_3=6\zeta_1^3a^3\lambda_1^2I\cos\Theta\frac{1}{\lambda_1^2-a^2\cos^2\Theta}$$
$$+\frac{2}{15}a^4\frac{\zeta_1}{\lambda_1^2-a^2\cos^2\Theta}(53\lambda_1^2+7a^2\cos^2\Theta),$$

and I is given by Eq. (23).

In addition, we assume that the relations

$$\lambda_1^2=\zeta_1^2+a^2,\quad \zeta_2=\alpha_3\zeta_1\left(\frac{3a^2}{2}+\zeta_1^2\right) \tag{30}$$

remain valid in the considered approximation.

Relations (28)–(30) provide the adiabatic evolution of the parameters a, ζ_1, and λ_1, respectively, in the presence of amplification-absorption terms and quintic nonlinearity as perturbation. For $a=0$ we can give the adiabatic evolution of the fundamental one-soliton solution of Eq. (3).

Exact solutions of Eqs. (28)–(30) can be obtained only when $\gamma_1=\gamma_2=\gamma_3=0$. In this case the field amplitude decreases for a purely absorbing medium ($\gamma_0<0$) or increases for a purely amplifying one ($\gamma_0>0$) according to the exponential laws $a(z)=a(0)\exp(\gamma_0 t)$ and $\zeta_1(z)=a(0)\exp(2\gamma_0 t)$. The pulse part increases or decreases faster than the cw part because of the factor of 2 in the exponential.

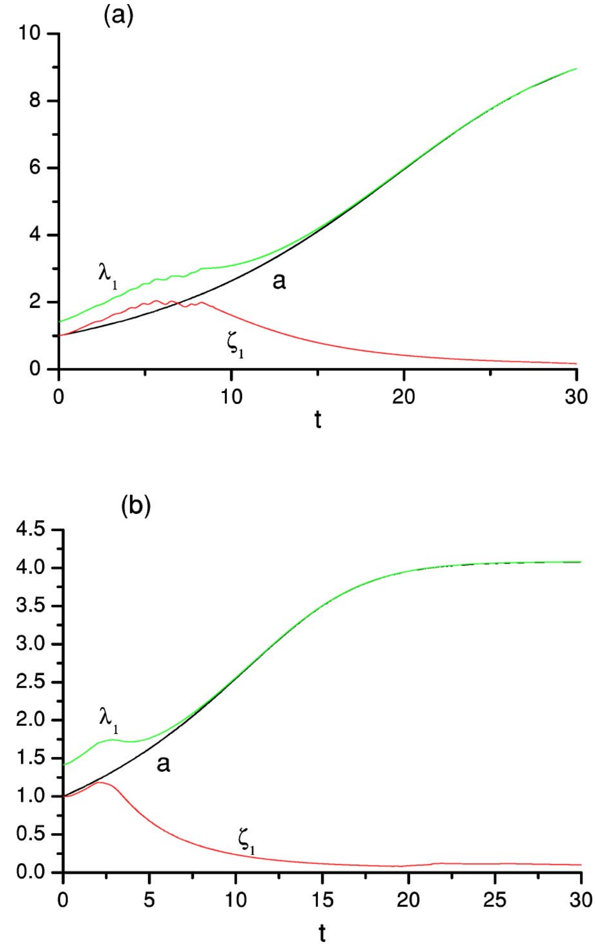

FIG. 3. (Color online) Adiabatic evolution of the parameters a, ζ_1, and λ_1 according to relations (28)–(30) for $\gamma_0=0.1$, $\gamma_1=0.01$, and (a) $\gamma_2=0.002$, $\gamma_3=0$, (b) $\gamma_2=0.002$, $\gamma_3=0.0003$.

For an amplifying medium ($\gamma_0>0$) with gain saturation, typical results are plotted in Fig. 3 by solving Eqs. (28)–(30) numerically. When the case of $\gamma_1>0$ and $\gamma_2=\gamma_3=0$ recovers the case of Fig. 5 in Ref. [38] where the cw part is uniformly amplified, while the pulse part amplifies rapidly before vanishing completely for large t ($\zeta_1\to 0$). The result is a growing cw asymptotic state that evolves according to $a(z)=a(0)\exp(\gamma_0 t)$. As depicted in Fig. 3(a), the evolution is similar for $\gamma_1>0$ and $\gamma_2>0$ but $\gamma_3=0$, except that the cw asymptotic state now saturates. And the small oscillations of a and ζ_1 can be seen clearly from Fig. 3(a), which is consistent with the results in Ref. [38]. However, when we take $\gamma_3\neq 0$ as shown in Fig. 3(b), these small ripples are eliminated. Therefore, we may infer that it is the quintic nonlinearity effect that makes the pulses more stable, which is important to the propagation of nonlinear pulses.

V. CONCLUSIONS

We have obtained two exact analytic solutions of the HNLS equation that describe (i) the homoclinic orbit of MI and (ii) soliton propagation on a continuous wave background by using the Darboux transformation. We also show how the higher-order terms influence these two solutions. We have also shown that the presence of higher-order terms, in general, changes the velocity of the solitary wave without changing its shape and that the sign of the third-order dispersion (α_3) determines the propagation direction of solitary waves. From the discussion above and from the relations $\alpha_2 = 2\mu^2\alpha_1$, $\alpha_4 = 6\mu^2\alpha_3$, and $\alpha_4 + \alpha_5 = 0$, we can see that it is the exact balance among the third-order dispersion, the self-steepening effect, and the delayed nonlinear response effect that make the pulse more stable for the second solution. Thus the compressed ultrafast pulses may be obtained by this method. Here we have analyzed how the amplification-absorption and quintic nonlinearity effects influence the second solution in the adiabatic approximation and have pointed out that these small ripples appearing in Fig. 3(a) as well as in Ref. [38] have been eliminated by introducing the quintic nonlinearity effect in Eq. (25). Therefore, we may infer that it is the quintic nonlinearity effect that makes the pulses more stable, which is important to the propagation of nonlinear pulses. The analytic result of the present paper will be helpful to know how analytical results can be applied to systems with realistic, nonintegrable higher-order terms.

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[1] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[2] L.F. Mollenauer and K. Smith, Opt. Lett. **13**, 675 (1988), and references therein.
[3] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 171 (1973).
[4] V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. **61**, 118 (1971) [Sov. Phys. JETP **34**, 62 (1972)].
[5] V.E. Zakharov and A.B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1974)].
[6] T. Kawata and H. Inoue, J. Phys. Soc. Jpn. **44**, 1722 (1978).
[7] Y.C. Ma, Stud. Appl. Math. **60**, 43 (1979).
[8] N.N. Akhmediev, V.M. Eleonskii, and N.E. Kulagin, Teor. Mat. Fiz. **72**, 183 (1987) [Theor. Math. Phys. **72**, 809 (1987)].
[9] H. Adachihara, D.W. McLaughlin, J.V. Moloney, and A.C. Newell, J. Math. Phys. **29**, 63 (1988).
[10] A. Hasegawa and Y. Kodama, Opt. Lett. **7**, 285 (1982).
[11] N.N. Akhmediev and S. Wabnitz, J. Opt. Soc. Am. B **9**, 236 (1992).
[12] N. Bélanger and P.-A. Bélanger, Opt. Commun. **124**, 301 (1996).
[13] Y. Kodama, J. Stat. Phys. **39**, 597 (1985).
[14] Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **23**, 510 (1987).
[15] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[16] K. Porsezian and K. Nakkeeran, Phys. Rev. Lett. **76**, 3955 (1996).
[17] J. Kim, Q.H. Park, and H.J. Shin, Phys. Rev. E **58**, 6746 (1998).
[18] M. Gedalin, T.C. Scott, and Y.B. Band, Phys. Rev. Lett. **78**, 448 (1997).
[19] S.L. Palacios, A. Guinea, J.M. Fernandez-Diaz, and R.D. Crespo, Phys. Rev. E **60**, R45 (1999).
[20] D. Mihalache, N. Truta, and L.C. Crasovan, Phys. Rev. E **56**, 1064 (1997).
[21] D. Mihalache, L. Torner, F. Moldoveanu, N.-C. Panoiu, and N. Truta, Phys. Rev. E **48**, 4699 (1993); J. Phys. A **26**, L757 (1993).
[22] D. Mihalache, N.-C. Panoiu, F. Moldoveanu, and D.-M. Baboiu, J. Phys. A **27**, 6177 (1994).
[23] Y.S. Kivshar and V.V. Afanasjev, Phys. Rev. A **44**, R1446 (1991).
[24] R. Radhakrishnan and M. Lakshmanan, Phys. Rev. E **54**, 2949 (1996).
[25] A. Mahalingam and K. Porsezian, Phys. Rev. E **64**, 046608 (2001).
[26] Z.H. Li, L. Li, H.P. Tian, and G.S. Zhou, Phys. Rev. Lett. **84**, 4096 (2000).
[27] W.P. Hong, Opt. Commun. **194**, 217 (2001).
[28] V.B. Matveev and M.A. Salli, *Darboux Transformations and Solitons*, Springer Series in Nonlinear Dynamics (Springer-Verlag, Berlin, 1991).
[29] Z.Y. Xu, L. Li, Z.H. Li, and G.S. Zhou, Opt. Commun. **210**, 375 (2002).
[30] R. Hirota, J. Math. Phys. **14**, 805 (1973).
[31] R.S. Tasgal and M.J. Potasek, J. Math. Phys. **33**, 1208 (1992).
[32] B.K. Som, M.R. Gupta, and B. Dasgupta, J. Phys. Soc. Jpn. **47**, 1296 (1979).
[33] S. Trillo and S. Wabnitz, Opt. Lett. **16**, 986 (1991).
[34] See, for example, M.J. Ablowitz and P.A. Clarkson, *Soliton, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).
[35] A. Hasegawa and W.F. Brinkman, IEEE J. Quantum Electron. **QE-16**, 694 (1980).
[36] K. Tai, A. Tomita, and A. Hasegawa, Phys. Rev. Lett. **56**, 135 (1986).
[37] N.N. Akhmediev, V.M. Eleonskii, and N.E. Kulagin, Sov. Phys. JETP **62**, 894 (1985).
[38] L. Gagnon, J. Opt. Soc. Am. B **10**, 469 (1993).